

MEAN DIMENSION AND A SHARP EMBEDDING THEOREM: EXTENSIONS OF APERIODIC SUBSHIFTS

YONATAN GUTMAN & MASAKI TSUKAMOTO

ABSTRACT. We show that if (X, T) is an extension of an aperiodic subshift (a subsystem of $(\{1, 2, \dots, l\}^{\mathbb{Z}}, \text{shift})$ for some $l \in \mathbb{N}$) and has mean dimension $\text{mdim}(X, T) < \frac{D}{2}$ ($D \in \mathbb{N}$), then it embeds equivariantly in $(([0, 1]^D)^{\mathbb{Z}}, \text{shift})$. The result is sharp. If (X, T) is an extension of an aperiodic zero-dimensional system then it embeds equivariantly in $(([0, 1]^{D+1})^{\mathbb{Z}}, \text{shift})$.

1. INTRODUCTION

Let (X, T) be a topological dynamical system (t.d.s), i.e., X is a compact metric space and $T : X \rightarrow X$ is a homeomorphism. Let D be a positive integer, and let $[0, 1]^D$ be the D -dimensional unit cube. Let $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$ be the shift on $[0, 1]^D$ i.e., $\sigma(x)_n \triangleq x_{n+1}$ for $x = (x_n)_{n \in \mathbb{Z}} \in ([0, 1]^D)^{\mathbb{Z}}$. We study the following problem in this paper:

Problem 1.1. When does there exist an embedding $\phi : (X, T) \rightarrow (([0, 1]^D)^{\mathbb{Z}}, \sigma)$? Here ϕ is called an (equivariant) embedding if ϕ is a topological embedding and $\phi T = \sigma \phi$.

Jaworski [Jaw74] proved that if (X, T) is a finite dimensional system and is aperiodic (i.e., it has no periodic points) then (X, T) can be embedded into the system $([0, 1]^{\mathbb{Z}}, \sigma)$ (See also Auslander [Aus88, Chapter 13, Theorem 9]). Clearly if a system has periodic points, this may constitute an obstruction to embedding into $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$. To quantify this introduce an invariant called *periodic dimension* $\text{perdim}(X, T) = \sup_{n \in \mathbb{N}} \frac{\dim(P_n)}{n}$, where P_n denotes the set of points of period $\leq n$. One readily checks that a necessary condition for embedding into $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$ is $\text{perdim}(X, T) \leq D$. In [Gut12b] it is shown that if (X, T) is a finite dimensional and $\text{perdim}(X, T) < \frac{D}{2}$, then (X, T) can be embedded into $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$.

For infinite dimensional systems the situation is very different. Lindenstrauss-Weiss [LW00, Proposition 3.3] constructed a minimal infinite dimensional system which cannot be embedded into the system $([0, 1]^{\mathbb{Z}}, \sigma)$ by using the theory of mean dimension.

Mean dimension (denoted by $\text{mdim}(X, T)$) is a topological invariant of dynamical systems introduced by Gromov [Gro99] and systematically investigated in [LW00]. The

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explicit formula appears at the end of the Preliminaries Section. The mean dimension of the system $\left(\left([0, 1]^D\right)^{\mathbb{Z}}, \sigma\right)$ is equal to D . If a system (X, T) can be embedded into another system (X', T') , then $\text{mdim}(X, T) \leq \text{mdim}(X', T')$. So if (X, T) can be embedded into $\left(\left([0, 1]^D\right)^{\mathbb{Z}}, \sigma\right)$ then we must have $\text{mdim}(X, T) \leq D$. Hence the mean dimension is also an obstruction to embedding a system into $\left(\left([0, 1]^D\right)^{\mathbb{Z}}, \sigma\right)$.

Surprisingly Lindenstrauss [Lin99, Theorem 5.1] proved the following partial converse:

Theorem 1.2. *If (X, T) is an extension of an aperiodic minimal system and satisfies $\text{mdim}(X, T) < D/36$ then (X, T) can be embedded into the system $\left(\left([0, 1]^D\right)^{\mathbb{Z}}, \sigma\right)$.*

In [Gut12a] it is shown one can replace the assumption of an aperiodic minimal system with the assumption of an aperiodic factor with a countable number of minimal subsystems or an aperiodic finite dimensional factor. In [Gut12b] the following theorem is proven:

Theorem 1.3. *If (X, T) is an extension of an aperiodic finite dimensional system and satisfies $\text{mdim}(X, T) < D/16$ then (X, T) can be embedded into the system $\left(\left([0, 1]^{D+1}\right)^{\mathbb{Z}}, \sigma\right)$.*

It is clear that both mean dimension and periodic dimension are obstructions for embedding. Recently Lindenstrauss and Tsukamoto conjectured that these are the only obstructions. Sacrificially Conjecture 1.2 of [LT12] states:

Conjecture 1.4. *If $\text{mdim}(X, T) < \frac{D}{2}$ and $\text{perdim}(X, T) < \frac{D}{2}$ then (X, T) can be embedded into the system $\left(\left([0, 1]^D\right)^{\mathbb{Z}}, \sigma\right)$.*

Our goal in this paper is to establish this conjecture in a special (infinite-dimensional aperiodic) case and show this result is sharp. As mentioned before [Gut12b] contains a proof that conjecture is true if X is finite dimensional.

Throughout this paper we use the following notation: For a map $f : X \rightarrow [0, 1]^D$ we define $I_f : X \rightarrow \left([0, 1]^D\right)^{\mathbb{Z}}$ by $I_f(x) \triangleq (f(T^n x))_{n \in \mathbb{Z}}$. Our main result is the following:

Theorem 1.5. *Let D be a positive integer. Let (Z, S) be an aperiodic zero dimensional system, and let $\pi : (X, T) \rightarrow (Z, S)$ be an extension of (Z, S) . If $\text{mdim}(X, T) < D/2$, then there exists a continuous map $f : X \rightarrow [0, 1]^D$ such that*

$$(I_f, \pi) : X \rightarrow \left([0, 1]^D\right)^{\mathbb{Z}} \times Z, \quad x \mapsto (I_f(x), \pi(x)),$$

is an embedding. Indeed such continuous maps $f : X \rightarrow [0, 1]^D$ form a comeagre subset of $C(X, [0, 1]^D)$ (the space of continuous maps from X to $[0, 1]^D$).

Example 1.6. Let (X, T) be an arbitrary dynamical system, and let (Z, S) be a zero dimensional system. As $\text{mdim}(X \times Z, T \times S) \leq \text{mdim}(X, T) + \text{mdim}(Z, S)$ ([LW00, Proposition 2.8]), then the product system $(X, T) \times (Z, S)$ is an extension of (Z, S) whose mean dimension is equal to the mean dimension of (X, T) .

Theorem 1.5 enables us to give a partial solution to Problem 1.1 by using:

Corollary 1.7. *Let D be a positive integer, and let (X, T) be an extension of an aperiodic zero dimensional system. If $\text{mdim}(X, T) < D/2$, then (X, T) can be embedded into $\left([0, 1]^{D+1}, \sigma\right)$.*

Proof. If (Z, S) is a zero dimensional system, then (Z, S) can be embedded into $\left([0, 1]^{\mathbb{Z}}, \sigma\right)$. Indeed there exists a topological embedding $g : Z \rightarrow [0, 1]$. Hence $I_g : (Z, S) \rightarrow \left([0, 1]^{\mathbb{Z}}, \sigma\right)$ gives an embedding of the system. Therefore the product system $\left(\left([0, 1]^D\right)^{\mathbb{Z}}, \sigma\right) \times (Z, S)$ can be embedded into the system $\left(\left([0, 1]^{D+1}\right)^{\mathbb{Z}}, \sigma\right)$. Corollary 1.7 now follows from Theorem 1.5. \square

In the context of the previous corollary, it should be noted that for \mathbb{Z}^k -extensions of aperiodic zero dimensional systems (\mathbb{Z}^k, X) , it was shown in [Gut11] that there exist constants $C(k) > 0$, so that (\mathbb{Z}^k, X) embeds in $\left(\left([0, 1]^{\lfloor C(k)\text{mdim}(\mathbb{Z}^k, X) \rfloor + 1}\right)^{\mathbb{Z}^k}, \mathbb{Z}^k - \text{shift}\right)$.

If we assume a stronger condition on the factor (Z, S) , then we can get a sharp result as follows. Let l be a positive integer, and let $(\{1, 2, \dots, l\}^{\mathbb{Z}}, \text{shift})$ be the shift on the alphabet $\{1, 2, \dots, l\}$. A subsystem (closed shift-invariant subset) of such a t.d.s is called a *subshift* (see [Wil04]).

Corollary 1.8. *Let D be a positive integer, and let (X, T) be an extension of an aperiodic subsystem of $(\{1, 2, \dots, l\}^{\mathbb{Z}}, \text{shift})$. If $\text{mdim}(X, T) < D/2$, then (X, T) can be embedded into $\left([0, 1]^D, \sigma\right)$.*

Proof. Suppose that (Z, S) is an aperiodic subsystem of $(\{1, 2, \dots, l\}^{\mathbb{Z}}, \text{shift})$ and that (X, T) is an extension of (Z, S) . Since Z is zero dimensional, Theorem 1.5 implies that (X, T) can be embedded into the system $\left(\left([0, 1]^D\right)^{\mathbb{Z}}, \sigma\right) \times (Z, S)$. From the assumption the latter is a subsystem of

$$(1.1) \quad \left([0, 1]^D \times \{1, 2, \dots, l\}^{\mathbb{Z}}, \text{shift}\right).$$

The space $[0, 1]^D \times \{1, 2, \dots, l\}$ can be topologically embedded into $[0, 1]^D$. Thus the above system (1.1) can be embedded into $\left([0, 1]^D, \sigma\right)$. \square

Corollary 1.8 is analogous to the following classical result in dimension theory ([HW41, p. 56, Theorem V 2]): If a compact metric space X satisfies $\dim X < D/2$, then X can be topologically embedded into $[0, 1]^D$. The following proposition shows that the condition $\text{mdim}(X, T) < D/2$ in Corollary 1.8 is optimal.

Proposition 1.9. *Let (Z, S) be a zero dimensional system. For any positive integer D , there exists an extension (X, T) of (Z, S) such that $\text{mdim}(X, T) = D/2$ and (X, T) cannot be embedded into $\left([0, 1]^D, \sigma\right)$.*

Indeed this proposition is a corollary of a stronger result (Proposition 4.2) given in Section 4.

2. PRELIMINARIES

In this section we present some basic facts on mean dimension theory. For more details, see Gromov [Gro99] and Lindenstrauss-Weiss [LW00]. Let (X, d) be a compact metric space, Y be a topological space and $f : X \rightarrow Y$ be a continuous map. f is called an ε -embedding for $\varepsilon > 0$ if $\text{diam}(f^{-1}(y)) < \varepsilon$ for every $y \in Y$. We define $\text{Widim}_\varepsilon(X, d)$ as the minimum of $n \geq 0$ such that there exist a n -dimensional polytope P and an ε -embedding $f : X \rightarrow P$. Equivalently, $\text{Widim}_\varepsilon(X, d)$ is the minimum $n \geq 0$ such that there exists an open covering $\{U_i\}_{i=1}^N$ of X satisfying $\text{diam}(U_i) < \varepsilon$ ($1 \leq i \leq N$) and $U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_{n+2}} = \emptyset$ for all $1 \leq i_1 < i_2 < \cdots < i_{n+2} \leq N$ (i.e. the order of $\{U_i\}_{i=1}^N$ is at most n). $\text{Widim}_\varepsilon(X, d)$ is monotone non-decreasing as $\varepsilon \rightarrow 0$.

The following is a key property of $\text{Widim}_\varepsilon(X, d)$ in the study of our embedding problem:

Lemma 2.1. *Let (X, d) be a compact metric space, and let $f : X \rightarrow [0, 1]^M$ be a continuous map. Suppose that positive numbers δ and ε satisfy the following condition:*

$$d(x, y) < \varepsilon \Rightarrow \|f(x) - f(y)\|_\infty < \delta.$$

(Here $\|\cdot\|_\infty$ is the ℓ^∞ -norm.) Under this condition, if $\text{Widim}_\varepsilon(X, d) < M/2$ then there exists an ε -embedding $g : X \rightarrow [0, 1]^M$ satisfying $\sup_{x \in X} \|f(x) - g(x)\|_\infty < \delta$.

Proof. Set $a \triangleq \text{Widim}_\varepsilon(X, d)$. Let $X = \bigcup_{i=1}^N U_i$ be an open covering of order a satisfying $\text{diam}(U_i) < \varepsilon$. Fix $x_i \in U_i$, and let $\{\varphi_i\}_{i=1}^N$ be a partition of unity satisfying $\text{supp } \varphi_i \subset U_i$. From the assumption on δ and ε , we have

$$s \triangleq \sup_{x \in \text{supp } \varphi_i} \|f(x) - f(x_i)\|_\infty < \delta.$$

Since $2a < M$, we can choose $u_i \in [0, 1]^M$ ($1 \leq i \leq N$) satisfying

$$\|u_i - f(x_i)\|_\infty < \delta - s$$

and the following condition: If there are $J, K \subset \{1, 2, \dots, N\}$ with $|J|, |K| \leq a + 1$, $\{\lambda_i\}_{i \in J} \in (\mathbb{R} \setminus \{0\})^{|J|}$ and $\{\mu_i\}_{i \in K} \in (\mathbb{R} \setminus \{0\})^{|K|}$ such that

$$\sum_{i \in J} \lambda_i = \sum_{i \in K} \mu_i = 1, \quad \sum_{i \in J} \lambda_i u_i = \sum_{i \in K} \mu_i u_i,$$

then $J = K$ and $\lambda_i = \mu_i$ for all $i \in J = K$. (The existence of such $\{u_i\}_{i=1}^N$ follows from the fact that one can choose almost surely (w.r.t Lebesgue measure) in $([0, 1]^M)^N$, $\{u_i\}_{i=1}^N$ to be affinely independent, see [Gut12b, Appendix]).

We define $g : X \rightarrow [0, 1]^M$ by setting $g(x) \triangleq \sum_{i=1}^N \varphi_i(x) u_i$. We have $g(x) - f(x) = \sum_i \varphi_i(x) (u_i - f(x))$. If $\varphi_i(x) \neq 0$ then from the definition of u_i

$$\|u_i - f(x)\|_\infty \leq \|u_i - f(x_i)\|_\infty + \|f(x_i) - f(x)\|_\infty < \delta.$$

Hence $\|g(x) - f(x)\|_\infty < \delta$ for all $x \in X$.

If $g(x) = g(y)$ for some $x, y \in X$, then the choice of $\{u_i\}$ implies that there exists $1 \leq i \leq N$ satisfying $\varphi_i(x) = \varphi_i(y) > 0$. Hence $x, y \in U_i$. Then $d(x, y) \leq \text{diam}(U_i) < \varepsilon$. This shows that g is an ε -embedding. \square

Suppose that $T : X \rightarrow X$ is a homeomorphism. For two integers $a < b$ we define a new distance d_a^b by setting $d_a^b(x, y) \triangleq \max_{a \leq i \leq b} d(T^i x, T^i y)$. We define the mean dimension $\text{mdim}(X, T)$ by

$$\text{mdim}(X, T) \triangleq \sup_{\varepsilon > 0} \left(\lim_{n \rightarrow +\infty} \frac{\text{Widim}_\varepsilon(X, d_0^{n-1})}{n} \right).$$

This limit always exists. (The limit value can be ∞ .)

3. PROOF OF THEOREM 1.5

In this section we prove Theorem 1.5. The basic structure of the proof is the same as Lindenstrauss [Lin99, Section 5]. Lindenstrauss proved Theorem 1.2 by using a Rohlin-tower like lemma [Lin99, Lemma 3.3]. The proof of this tower lemma uses the assumption that (X, T) is an extension of an infinite minimal system. Here we assume that (X, T) is an extension of an aperiodic zero dimensional system. This assumption implies a much stronger tower lemma (Lemma 3.3), and thus we are able to prove a sharp result for the embeddedability of (X, T) . Throughout this section, (X, d) is a compact metric space and $T : X \rightarrow X$ is a homeomorphism such that there exist an aperiodic zero dimensional system (Z, S) and a factor map $\pi : (X, T) \rightarrow (Z, S)$. Moreover we assume that $\text{mdim}(X, T) < D/2$ for a positive integer D . For simplicity of the notation, we set $K \triangleq [0, 1]^D$.

For $\eta > 0$, we define a subset $A(\eta)$ of $C(X, K)$ by

$$A(\eta) \triangleq \{f \mid (I_f, \pi) : X \rightarrow K^\mathbb{Z} \times Z \text{ is an } \eta\text{-embedding with respect to } d.\}.$$

It is easy to see that $A(\eta)$ is an open set. We want to prove the following proposition:

Proposition 3.1. *For any continuous $f : X \rightarrow K$, $\delta > 0$ and $\eta > 0$ there exists a continuous map $g : X \rightarrow K$ such that*

- (i) $\|f(x) - g(x)\|_\infty < \delta$ for all $x \in X$.
- (ii) $(I_g, \pi) : X \rightarrow K^\mathbb{Z} \times Z$ is an η -embedding with respect to the distance d .

Assume this proposition. Then $A(\eta)$ is dense in $C(X, K)$. The Baire Category Theorem implies that

$$\bigcap_{n=1}^{\infty} A(1/n)$$

is comeagre and hence dense. Then every f in this set gives a desired solution. So the main problem is to prove Proposition 3.1.

3.1. Representation lemma. Recall that (Z, S) is an aperiodic zero dimensional system, and that the zero dimensionality implies that clopen subsets (closed and open subsets) form a basis of the topology of Z .

We start by a representation lemma for (Z, S) . As it similar to the well known concept of a *special automorphism* in measured dynamics (see [Ao]) we call such a representation a *special topological dynamical system*.

Definition 3.2. Let B be a zero dimensional compact metric space. Let $T_B : B \rightarrow B$ be an automorphism. Let $h : B \rightarrow \mathbb{Z}_{\geq 0}$ be continuous. Define $Q = \{(b, z) \mid b \in B, 0 \leq z < h(b)\} \subset B \times \mathbb{Z}_{\geq 0}$. As h is bounded this is a compact metric zero dimensional space. Define $T : Q \rightarrow Q$ by:

$$T(b, z) = \begin{cases} (b, z + 1) & z + 1 < h(b) \\ (T_B(b), 0) & z + 1 = h(b) \end{cases}$$

Clearly T is an automorphism. B is referred to as the *base* of Q . $h : B \rightarrow \mathbb{Z}_{\geq 0}$ is referred to as the *roof function* of Q . (Q, T) is referred to as a *special topological dynamical system* and it is said to be *induced* by (B, T_B, h) .

Lemma 3.3. For any positive integer N there exists a clopen set $B \subset Z$, a continuous function $h : B \rightarrow R_N \triangleq \{N+1, \dots, 2N+1\}$ and a continuous automorphism $T_B : B \rightarrow B$ so that (Z, S) is isomorphic (as a dynamical system) to a special t.d.s (Q_N, T') induced by (B, T_B, h) . We denote this isomorphism by $(b, n) : Z \rightarrow Q_N \subset B \times \mathbb{Z}_{\geq 0}$.

Notation: We say such B, T_B, h are associated with N .

Proof. There exist clopen sets U_1, \dots, U_m such that $X = U_1 \cup \dots \cup U_m$ and $U_i \cap S^k U_i = \emptyset$ for all $1 \leq i \leq m$ and $1 \leq k \leq N$. We define clopen sets V_l ($1 \leq l \leq m$) by $V_1 \triangleq U_1$ and $V_{l+1} \triangleq V_l \cup \left(U_{l+1} \setminus \bigcup_{i=-N}^N S^i V_l \right)$. We can directly check the following two conditions by using the induction on $l = 1, \dots, m$:

$$V_l \cap S^k V_l = \emptyset \quad (1 \leq l \leq m, 1 \leq k \leq N),$$

$$U_1 \cup \dots \cup U_l \subset \bigcup_{k=-N}^N S^k V_l \quad (1 \leq l \leq m).$$

Then the clopen set $B \triangleq V_m$ satisfies $B \cap S^k B = \emptyset$ ($1 \leq k \leq N$) and $X = \bigcup_{k=1}^{2N+1} S^k B$. Define the continuous function $h : B \rightarrow R_N \triangleq \{N+1, \dots, 2N+1\}$ by $h(b) = \min\{n \geq 1 \mid S^n b \in B\}$. Clearly $N < h(b) \leq 2N+1$. Define $T_B : B \rightarrow B$ by $T_B b = S^{h(b)} b$. T_B is clearly continuous as h is locally constant. T_B is invertible as $T_B^{-1} b = S^{-n(b)} b$ where $n(b) = \min\{l \geq 1 \mid S^{-l} b \in B\}$ and it clearly holds $h(T_B^{-1} b) = n(b)$. Notice $n : X \rightarrow \{1, \dots, 2N+1\}$ extends continuously to all of X by the same formula. Let (Q_N, T') be the a special t.d.s induced by (B, T_B, h) . Define $b : Z \rightarrow B$ by $b(z) = S^{-n(z)} z$. Note $(b, n) : (Z, S) \rightarrow (Q_N, T')$ is indeed an equivariant isomorphism. \square

3.2. Proof of Proposition 3.1. Throughout this subsection we fix a continuous function $f : X \rightarrow K$ (recall: $K = [0, 1]^D$) and positive numbers δ and η . Fix $0 < \varepsilon < \eta$ such that

$$(3.1) \quad d(x, y) < \varepsilon \Rightarrow \|f(x) - f(y)\|_\infty < \delta.$$

Since $\text{mdim}(X, T) < D/2$, we can take a positive integer L such that for every $k \geq L$

$$(3.2) \quad \frac{\text{Widim}_\varepsilon(X, d_0^{k-1})}{k} < \frac{D}{2}.$$

Applying Lemma 3.3 to (Z, S) , let $Q = Q_L(B, T_B, h)$ be the special t.d.s induced by (B, T_B, h) associated with L . Let $B_k = (h \circ \pi)^{-1}\{k\}$, $k \in R_L$. These are clopen sets. We consider the function $(I_f|_0^{k-1})|_{B_k}(x) \triangleq (f(x), f(Tx), \dots, f(T^{k-1}x)) \in K^k$ on the metric space (B_k, d_0^{k-1}) . Notice that by (3.1) $d_0^{k-1}(x, y) < \varepsilon$ implies $\|I_f|_0^{k-1}(x) - I_f|_0^{k-1}(y)\|_\infty < \delta$. By Lemma 2.1 there is an ε -embedding $F_k : (B_k, d_0^{k-1}) \rightarrow K^k$ satisfying

$$(3.3) \quad \sup_{x \in B_k} \|F_k(x) - I_f(x)|_0^{k-1}\|_\infty < \delta,$$

Define $g : X \rightarrow K$ by $g(T^l x) \triangleq F_k(x)|_l$ for $x \in B_k$ and $0 \leq l < k$. Clearly g is continuous and by (3.3) $\sup_{x \in X} \|g(x) - f(x)\|_\infty < \delta$. Now assume $(I_g(x), \pi(x)) = (I_g(y), \pi(y))$. As $\pi(x) = \pi(y)$ we have $b(\pi(x)) = b(\pi(y))$ and $n \triangleq n(\pi(x)) = n(\pi(y))$. Conclude there exists $k \in R_L$ so that with $0 \leq n < k$ and $T^{-n}x, T^{-n}y \in B_k$. From $I_g(T^{-n}x)|_0^{k-1} = I_g(T^{-n}y)|_0^{k-1}$ we conclude $F_k(T^{-n}x) = F_k(T^{-n}y)$. As $F_k : (B_k, d_0^{k-1}) \rightarrow K^k$ is an ε -embedding we conclude $d_0^{k-1}(T^{-n}x, T^{-n}y) < \varepsilon$ which implies $d(x, y) < \varepsilon$. \square

4. PROOF OF PROPOSITION 1.9

The argument in this section is a slight modification of the argument of Lindenstrauss-Tsukamoto [LT12, Lemma 3.3]. Let Y be the triod graph (the graph of the shape “Y”). Let d_Y be the graph distance on Y where all three edges have length 1. Consider Y^n with the distance $d_{\ell^\infty}(x, y) \triangleq \max_{1 \leq i \leq n} d_Y(x, y)$. The following fact is proved in [LT12, Proposition 2.5] by using the method of Matoušek [Mat03]:

Lemma 4.1. *For $0 < \varepsilon < 1$ there does not exist an ε -embedding from the space (Y^n, d_{ℓ^∞}) to \mathbb{R}^{2n-1} .*

Let D be a positive integer. In [LT12, Section 3] a compact metric space (X, d) with a homeomorphism $T : X \rightarrow X$ satisfying the following two conditions was constructed:

- (i) $\text{mdim}(X, T) = D/2$.
- (ii) There exist sequences of positive integers $\{c_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ such that

$$\lim_{n \rightarrow \infty} b_n = \infty, \quad \lim_{n \rightarrow \infty} \left(c_n - \frac{Db_n}{2} \right) = \infty,$$

and so that there exists a distance-increasing continuous map from $(Y^{c_n}, d_{\ell^\infty})$ to $(X, d_0^{b_n-1})$ for every $n \geq 1$ ([LT12, Lemma 3.1]). (A map $f : (Y^{c_n}, d_{\ell^\infty}) \rightarrow (X, d_0^{b_n-1})$ is said to be distance-increasing if $d_0^{b_n-1}(f(x), f(y)) \geq d_{\ell^\infty}(x, y)$.)

Moreover one can assume that (X, T) is minimal. But this is not used below. (Remark: In the notation of [LT12, Section 3], we have $c_n = Db_n \prod_{k=1}^n (1 - p_k)$.)

Proposition 1.9 follows from the consideration in Example 1.6 and the following proposition. (We set $K = [0, 1]^D$ as in Section 3.)

Proposition 4.2. *For any dynamical system (Z, S) , the product system $(X, T) \times (Z, S)$ cannot be embedded into $(K^{\mathbb{Z}}, \sigma)$.*

Proof. Let d_Z be the distance on Z . We define a distance on $X \times Z$ by $\text{dist}((x, z), (x', z')) \triangleq \max(d(x, x'), d_Z(z, z'))$. Fix $p \in Z$. The map

$$X \rightarrow X \times Z, \quad x \mapsto (x, p),$$

gives an isometric embedding from (X, d_0^{n-1}) to $(X \times Z, \text{dist}_0^{n-1})$ for every $n \geq 1$. From property (ii) of (X, T) , there exists a distance-increasing continuous map from $(Y^{c_n}, d_{\ell^\infty})$ to $(X \times Z, \text{dist}_0^{b_n-1})$ for every $n \geq 1$. From this point on one can use the proof of [LT12, Section 3] with X replaced by $X \times Z$ verbatim.

Suppose that there exists an embedding $f : (X \times Z, T \times S) \rightarrow (K^{\mathbb{Z}}, \sigma)$. Let d' be a distance on $K^{\mathbb{Z}}$. There exists $\varepsilon > 0$ such that if $x, y \in X \times Z$ satisfy $d'(f(x), f(y)) < \varepsilon$ then $\text{dist}(x, y) < 1/2$. Then, for any $N \geq 1$, $d'^{N-1}(f(x), f(y)) < \varepsilon$ implies $\text{dist}_0^{N-1}(x, y) < 1/2$.

We can take $L = L(\varepsilon) > 0$ such that if $x, y \in K^{\mathbb{Z}}$ satisfy $x_n = y_n$ ($-L \leq n \leq L$) then $d'(x, y) < \varepsilon$. This implies that, for any $N \geq 1$, if $x, y \in K^{\mathbb{Z}}$ satisfies $x_n = y_n$ ($-L \leq n \leq N + L - 1$) then we have $d'^{N-1}(x, y) < \varepsilon$. Let $\pi_{[-L, N+L-1]} : K^{\mathbb{Z}} \rightarrow K^{\{-L, -L+1, \dots, N+L-1\}}$, $x \mapsto (x_n)_{-L \leq n \leq N+L-1}$, be the projection. Then, from the above discussions, $\pi_{[-L, N+L-1]} \circ f : (X \times Z, \text{dist}_0^{N-1}) \rightarrow K^{\{-L, -L+1, \dots, N+L-1\}}$ is a $(1/2)$ -embedding. Since there exists a distance-increasing continuous map from $(Y^{c_n}, d_{\ell^\infty})$ to $(X \times Z, \text{dist}_0^{b_n-1})$, we conclude that there exists a $(1/2)$ -embedding from $(Y^{c_n}, d_{\ell^\infty})$ to $K^{\{-L, -L+1, \dots, b_n+L-1\}} = [0, 1]^{D(b_n+2L)}$ for every $n \geq 1$. Applying Lemma 4.1 to this situation, we get $2c_n \leq D(b_n + 2L)$. Hence

$$c_n - \frac{Db_n}{2} \leq LD.$$

This contradicts the condition (ii): $\lim_{n \rightarrow \infty} (c_n - Db_n/2) = \infty$. \square

REFERENCES

- [Ao] D.V. Anosov (originator). Special automorphism. Encyclopedia of Mathematics (online).
- [Aus88] Joseph Auslander. *Minimal flows and their extensions*, volume 153 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1988. Notas de Matemática [Mathematical Notes], 122.
- [Gro99] Misha Gromov. Topological invariants of dynamical systems and spaces of holomorphic maps. I. *Math. Phys. Anal. Geom.*, 2(4):323–415, 1999.
- [Gut11] Yonatan Gutman. Embedding \mathbb{Z}^k -actions in cubical shifts and \mathbb{Z}^k -symbolic extensions. *Ergodic Theory Dynam. Systems*, 31(2):383–403, 2011.
- [Gut12a] Yonatan Gutman. Dynamical embedding in cubical shifts and the topological Rokhlin and small boundary properties. Preprint, 2012.

- [Gut12b] Yonatan Gutman. Mean dimension and Jaworski-type theorems. Preprint, 2012.
- [HW41] Witold Hurewicz and Henry Wallman. *Dimension Theory*. Princeton Mathematical Series, v. 4. Princeton University Press, Princeton, N. J., 1941.
- [Jaw74] A. Jaworski. *The Kakutani-Bebutov theorem for groups*. Ph.D. dissertation. University of Maryland, 1974.
- [Lin99] Elon Lindenstrauss. Mean dimension, small entropy factors and an embedding theorem. *Inst. Hautes Études Sci. Publ. Math.*, 89(1):227–262 (2000), 1999.
- [LT12] Elon Lindenstrauss and Masaki Tsukamoto. Mean dimension and an embedding problem: an example. Preprint, 2012.
- [LW00] Elon Lindenstrauss and Benjamin Weiss. Mean topological dimension. *Israel J. Math.*, 115:1–24, 2000.
- [Mat03] Jiří Matoušek. *Using the Borsuk-Ulam theorem*. Universitext. Springer-Verlag, Berlin, 2003. Lectures on topological methods in combinatorics and geometry, Written in cooperation with Anders Björner and Günter M. Ziegler.
- [Wil04] Susan G. Williams. Introduction to symbolic dynamics. In *Symbolic dynamics and its applications*, volume 60 of *Proc. Sympos. Appl. Math.*, pages 1–11. Amer. Math. Soc., Providence, RI, 2004.

Yonatan Gutman, Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland.

E-mail address: `y.gutman@impan.pl`

Masaki Tsukamoto

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

E-mail address: `tukamoto@math.kyoto-u.ac.jp`